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JORDAN DOMAINS AND THE UNIVERSAL TEICHMÜLLER SPACE

BY

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ABSTRACT. Let L denote the lower half plane and let $B(L)$ denote the Banach space of analytic functions f in L with $\|f\|_L < \infty$, where $\|f\|_L$ is the suprenum over $z \in L$ of the values $|f(z)|(\operatorname{Im} z)^2$. The universal Teichmüller space, T, is the subset of $B(L)$ consisting of the Schwarzian derivatives of conformal mappings of L which have quasiconformal extensions to the extended plane. We denote by J the set

$$\{S_f : f \text{ is conformal in } L \text{ and } f(L) \text{ is a Jordan domain}\},$$

which is a subset of $B(L)$ contained in the Schwarzian space S. In showing $S - \bar{T} \neq \varnothing$, Gehring actually proves $S - \bar{J} \neq \varnothing$. We give an example which demonstrates that $J - \bar{T} \neq \varnothing$.

1. Introduction. If D is a simply connected domain of hyperbolic type in $\overline{\mathbf{C}}$, then the hyperbolic metric in D is given by

$$\rho_D(z) = \frac{2|g'(z)|}{1 - |g(z)|^2}, \qquad z \in D,$$

where g is any conformal mapping of D onto the unit disk $\Delta = \{z : |z| < 1\}$. If f is a locally univalent meromorphic function in D, the Schwarzian derivative of f is given by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

at finite points of D which are not poles of f. The definition of S_f is extended to all of D by means of inversions. We let $B(D)$ denote the Banach space of Schwarzian derivatives of all such functions f in a fixed domain D for which the norm

$$\|S_f\|_D = \sup_{z \in D} |S_f(z)| \rho_D(z)^{-2}$$

is finite.

In the case that D is the lower half plane $L = \{z : \operatorname{Im} z < 0\}$ certain subsets of $B(L)$ are of special interest. We let

$$S = \{S_f : f \text{ is conformal in } L\},$$

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$$J = \{S_f \in S: f(L) \text{ is a Jordan domain}\},$$
$$T = \{S_f \in S: \partial f(L) \text{ is a quasicircle}\}.$$

T is called the universal Teichmüller space: it is known that T is open, S is closed and $T = \operatorname{Int}(S)$ (see [**1**, **3**]).

In [**4**], Gehring shows $S - \bar{T} \neq \varnothing$, but his proof actually gives $S - \bar{J} \neq \varnothing$. We will show that $J - \bar{T} \neq \varnothing$.

We recall the key result and construction in [**4**]. Let $a > 0$, and set

$$\beta = \{\pm ie^{(-a+i)t}: t \in (-\infty, \infty)\} \cup \{0, \infty\},$$
$$\gamma = \beta \cap \bar{\Delta}, \qquad \mathfrak{D} = \bar{\mathbf{C}} - \gamma.$$

THEOREM 1 (GEHRING). *If $a \in (0, 1/8\pi)$, then there exists $\delta = \delta(a) > 0$ such that if f is conformal in $\mathfrak{D}$ with $\|S_f\|_{\mathfrak{D}} \leqslant \delta$, then $\partial f(\mathfrak{D})$ is not a quasicircle.*

That $S - \bar{T} \neq \varnothing$ is an immediate corollary of Theorem 1 and the transformation law for the Schwarzian derivative, $S_{f \circ g} = (S_f \circ g)g'^2 + S_g$, which implies $\|S_{f \circ g} - S_g\|_L = \|S_f\|_{g(L)}$. Now let g be a conformal mapping of L onto $\mathfrak{D}$ and let h be a conformal mapping of L with $\|S_h - S_g\|_L < \delta$; then $f = h \circ g^{-1}$ is a conformal mapping of $\mathfrak{D}$ with $\|S_f\|_{\mathfrak{D}} < \delta$. By Theorem 1, $\partial f(\mathfrak{D}) = \partial h(L)$ is not a quasicircle; consequently, $S_h \notin T$ and $S_g \in S - \bar{T}$.

The crux of Gehring's argument is showing if $\|S_f\|_{\mathfrak{D}} \leqslant \delta$, and if $\mathfrak{D}_j$, $j = 1, 2$, denotes the component of $\mathfrak{D} - \beta$ containing α_j, where

$$\alpha_1 = \{e^{(-a+i)t}: t \in (0, \infty)\}, \qquad \alpha_2 = \{-z: z \in \alpha_1\},$$

then the mappings $f|\mathfrak{D}_j$ have the same limit as z tends to 0 on α_j. Thus $f(\mathfrak{D})$ is not even a Jordan domain, and it follows that $S - \bar{J} \neq \varnothing$. For what remains, we fix $a \in (0, 1/8\pi)$ and so fix γ, $\mathfrak{D}$, α_1 and α_2. Our aim is to establish the following result.

THEOREM 2. *There exists a Jordan domain D and a constant $d = d(a) > 0$ such that if f is conformal in D and $\|S_f\|_D \leqslant d$ then $\partial f(D)$ is not a quasicircle.*

COROLLARY. $J - \bar{T} \neq \varnothing$.

The Corollary follows from Theorem 2 in the same manner that $S - \bar{T} \neq \varnothing$ follows from Theorem 1.

We construct a candidate Jordan domain D whose boundary consists of a line with a countable number of spiral-like wrinkles in it: the wrinkles are Jordan arcs resembling γ. We show how to find the appropriate value of d using methods like those in [**4**], but the proof that D and d satisfy Theorem 2 requires a different argument. In this case, $\partial f(D)$ may be a Jordan curve for $\|S_f\|_D < d$, so we use normal families and a geometric characterization of quasicircles to show that $\partial f(D)$ is not a quasicircle.

2. Construction of the candidate domain. It is simplest to describe D by giving its complement. For this, we first construct a sequence of closed Jordan regions E_m with $\bar{\Delta} \supset E_1 \supset E_2 \supset \cdots$ and $\bigcap_{m=1}^{\infty} E_m = \gamma$, and then attach a translation of each E_m to the closed half plane $H = \{x + iy: y \leqslant -1\} \cup \{\infty\}$. More precisely, let $\sigma_m = (\pi/8)^m$,

$\tau_m = e^{-2\pi a m}$, and set $E_m = R_m \cup P_m$ where

$$P_m = \{e^{i\sigma}z\colon z \in \gamma, -\sigma_m \leqslant \sigma \leqslant \sigma_m\} \cup \{z\colon |z| \leqslant \tau_m\},$$
$$R_m = \{x + iy\colon |x| \leqslant \sin\sigma_m, -1 \leqslant y \leqslant -\cos\sigma_m\} - \Delta.$$

Let V denote the translation $V(z) = z + 8$ and set

$$D = \overline{\mathbf{C}} - \left(H \cup \bigcup_{m=1}^{\infty} V^m(E_m)\right).$$

To see that ∂D is a Jordan curve, we note that $\gamma_m = \partial D \cap \{x + iy\colon -4 \leqslant x - 8m < 4\}$ is a half-open Jordan arc from $-4 + 8m$ to $4 + 8m$ for $m = 1, 2, \ldots$, and that ∂D may be written as the union of pairwise disjoint components,

$$\partial D = \bigcup_{m=1}^{\infty} \gamma_m \cup (-\infty, 4) \cup \{\infty\}.$$

Another way to see that ∂D is a simple, closed curve in $\overline{\mathbf{C}}$ is to consider its image under the Möbius transformation $z \to (z + 2i)^{-1}$.

Throughout the proof of Theorem 2 we will refer to a sequence of domains D_m with $D_m \subset V^{-m}(D)$. Let A denote the open region

$$A = \{x + iy\colon y > 1\} \cup \{x + iy\colon |x| < 4, -1 < y \leqslant 1\}$$

and set

$$D_m = A - E_m, \quad m = 1, 2, \ldots, \qquad D_\infty = A - \gamma.$$

Note that $D_m \subset D_{m+1}$ for all m and $\bigcup_{m=1}^{\infty} D_m = D_\infty = A \cap \mathfrak{D}$. For each m, including $m = \infty$, and for $j = 1, 2$, we let $D_{m,j}$ denote the component of $D_m - \Gamma$ containing $D_m \cap \alpha_j$ where $\Gamma = \{iy\colon y > 1\}$.

Crucial to our argument is the fact that $\partial D_{m,j}$ is a K_0-quasicircle for some fixed $K_0 = K_0(a) \in (1, \infty)$ and for all values of m and j. We sketch the proof of this fact: the idea is to find, for each m, a K_0'-quasiconformal mapping F_m of $\overline{\mathbf{C}}$ which maps $D_{m,2}$ onto $A \cap \{z\colon \operatorname{Re} z < 0\}$. A similar construction yields a quasiconformal mapping G_m of $\overline{\mathbf{C}}$ with $G_m(D_{m,1}) = A \cap \{z\colon \operatorname{Re} z > 0\}$ and $K(G_m) \leqslant (a + 2/a)^2 K_0'$. Both $A \cap \{z\colon \operatorname{Re} z < 0\}$ and $A \cap \{z\colon \operatorname{Re} z > 0\}$ are K-quasidisks for some finite K, so our claim is established with $K_0 = K \cdot K_0' \cdot (a + 2/a)^2$.

The mappings F_m are compositions of three basic types of mappings, each of which is the identity mapping outside either a disk or a rectangle. First consider the $(a + 2/a)$-quasiconformal mapping h_a of $\overline{\mathbf{C}}$ which fixes 0 and ∞ and satisfies

$$h_a(re^{i\theta}) = r^a e^{i(\theta - \log r)}, \qquad r \in (0, \infty)$$

(see [**4**]). We use h_a^{-1} to define an $(a + 2/a)$-quasiconformal mapping h of $\overline{\mathbf{C}}$ which fixes every point outside Δ; namely,

$$h(z) = \begin{cases} z, & z \notin \Delta, \\ h_a^{-1}(z), & z \in \Delta. \end{cases}$$

We may take $G_\infty = F_\infty = h$, and although we must continue for finite m our task is simplified because $h(D_{m,2})$ is bounded by circular arcs and line segments.

The second function we use is a composition of the "quasiconformal foldings" described in [**6**, Lemma 13]. Briefly, if $r > 0$ and $\theta \in (0, \pi)$, $f[r, \theta; \phi]$ is a $\pi/(\pi - \theta)$-quasiconformal mapping of $\overline{\mathbf{C}}$ which maps the arc $\{re^{i\psi}: \psi \in [\phi - \theta/2, \phi + \theta/2]\}$ onto the line segment with the same endpoints, and fixes every point outside the disk whose boundary is orthogonal to $\{z: |z| = r\}$ at those endpoints. More precisely, $f[r, \theta; \phi]$ is the conjugation by a Möbius transformation of the mapping F which fixes 0 and ∞ and satisfies $F(re^{i\psi}) = re^{ig(\psi)}$ for $r > 0$ and $\psi \in [-\pi/2, 3\pi/2]$. We define g to be the continuous function which is linear on each of the intervals $[-\pi/2, \pi/2]$, $[\pi/2, \pi]$, $[\pi, 3\pi/2]$ and satisfies $g(-\pi/2) = -\pi/2$, $g(\pi/2) = \pi/2$, $g(\pi) = \pi - \theta/2$ and $g(3\pi/2) = 3\pi/2$. We set

$$f_m = f[r, \theta; \phi] \circ f[r, \theta; -\phi]$$

where $r = \tau_m^{1/a}$, $\theta = (\pi - 2\sigma_m)/2$ and $\phi = (3\pi + 2\sigma_m)/4$. Note that $K(f_m) \leqslant 4$ for all m.

The third type of mapping is guaranteed by the lemma below: it fixes every point outside a rectangle and maps a cross-cut of the rectangle onto the segment with the same endpoints.

LEMMA 1. *Let* $0 < y_1 < y_2$, $\alpha \in (0, \pi/2)$, *and suppose* f: $[x_1, x_2] \to [y_1, y_2]$ *is a piecewise differentiable function with* $f(x_1) = f(x_2) = y_1$ *and, for all* $x, x' \in [x_1, x_2]$,

$$|f(x) - f(x')| \leqslant |x - x'| \tan\alpha.$$

Then there exists a $(1 + k)/(1 - k)$*-quasiconformal mapping* g *of* $\overline{\mathbf{C}}$ *which maps every vertical line onto itself, fixes every point outside* $R = \{x + iy: x_1 < x < x_2, 0 < y < y_1 + y_2\}$, *and satisfies* $g(x + if(x)) = x + iy_1$ *for* $x \in (x_1, x_2)$, *where*

$$k = \left(1 - \frac{4}{4 + \tan^2\alpha}\left(\frac{y_1}{y_2}\right)^2\right)^{1/2}.$$

PROOF. An easy check shows the mapping g defined by

$$g(x + iy) = \begin{cases} x + iy, & x + iy \notin R, \\ x + i\left(y_1 + y_2 \dfrac{y - f(x)}{y_1 + y_2 - f(x)}\right), & x + iy \in R, y \geqslant f(x), \\ x + iyy_1/f(x), & x + iy \in R, y \leqslant f(x), \end{cases}$$

is a homeomorphism of $\overline{\mathbf{C}}$ fixing each point of $\overline{\mathbf{C}} - R$, mapping vertical lines onto themselves, and satisfying $g(x + if(x)) = x + iy_1$. One also computes that g is ACL in $\overline{\mathbf{C}}$ with $|g_{\bar{z}}| \leqslant k|g_z|$ almost everywhere.

We define g_m to be a mapping of the type in Lemma 1 which takes

$$\{ire^{i\sigma_m}: 1 - 2\sin(\sigma_m/2) \leqslant r \leqslant 1\} \cup \{ie^{i\theta}: 0 \leqslant \theta \leqslant \sigma_m\}$$

onto the segment from $i(1 - 2\sin(\sigma_m/2))e^{i\sigma_m}$ to i. Finally, we set

$$F_m = w_m \circ g_m \circ f_m \circ h, \qquad G_m = r \circ F_m \circ h^{-1} \circ r \circ h,$$

where r denotes reflection in the imaginary axis and w_m denotes another mapping of the type in Lemma 1. We take w_m to fix every point outside $[-2, 1] \times [-3, 1]$, to map horizontal lines onto themselves, and to take the arc of $A \cap \partial(g_m \circ f_m \circ h(D_{m,2}))$ from $-i - \sin\sigma_m$ to i onto the line segment from $-i$ to i. The definitions of τ_m and σ_m give uniform bonds for the dilatations of g_m and w_m; therefore, we obtain a uniform bound, $K_0'(a)$, for $K(F_m)$. Thus, by the definition of G_m, $K(G_m) \leqslant (a + 2/a)^2 K_0'(a)$, and our claim is established.

REMARK. Since $\partial D_{m,j}$ is a $K_0(a)$-quasicircle for all m and j, we are guaranteed the existence of $d_1 = d_1(a) > 0$ such that the following holds for every m (see Lemma 6 of [4]). If f is a conformal mapping of D_m and if $\|S_f\|_{D_m} \leqslant d_1$, then for $j = 1, 2$ the mapping $f_j = f \mid D_{m,j}$ has a K-quasiconformal extension g_j to $\overline{\mathbf{C}}$ with $K \leqslant (1 - c\|S_f\|_{D_m})^{-1}$ and $c = c(a)$. In this case f_j has a homeomorphic extension to $\overline{D_{m,j}}$ which we also denote by f_j. If $z \in \Gamma$ then $f_1(z) = f(z) = f_2(z)$, and the continuity of f_1 and f_2 implies $f_1(i) = f_2(i)$ and $f_1(\infty) = f_2(\infty)$. These two common values will be denoted $f(i)$ and $f(\infty)$, respectively.

3. A mapping property of D_∞. Our next step is to show that a conformal mapping of D_∞ with sufficiently small Schwarzian norm is fairly rigid. The first part of the following lemma gives the value of d for Theorem 2 and states that Theorem 1 holds for D_∞ and d in place of $\mathfrak{D}$ and δ. The second part gives an estimate we use in proving $f(D)$ is not a quasidisk when $\|S_f\|_D < d$.

LEMMA 2. *There exists $d = d(a) \in (0, d_1]$ such that whenever f is a conformal mapping of D_∞ with $\|S_f\|_{D_\infty} \leqslant d$, then $f(D_\infty)$ is not a Jordan domain. In fact, if $f_j = f \mid D_{\infty,j}$, then $f_1(0) = f_2(0)$.*

If, moreover, f fixes -1, -3 and ∞ then

$$|f(0) - f(i)| \geqslant 1/3 \tag{3.1}$$

where $f(0)$ denotes the common value of $f_1(0)$ and $f_2(0)$.

Before proving Lemma 2 we state three propositions that are analogues of Lemmas 7, 8 and 9 in [4], respectively. We prove only Proposition 3 since the proofs of the first two propositions are identical to those of the corresponding lemmas.

PROPOSITION 1. *For each $\eta > 0$ there exists $K_1 = K_1(\eta) \in (1, \infty)$ such that if g is a sense-preserving K_1-quasiconformal mapping of $\overline{\mathbf{C}}$ with $g(\infty) = \infty$ and if z_1 and z_2 are distinct points in $\mathbf{C}$, then*

$$\left| \frac{g(z) - g(z_2)}{g(z_1) - g(z_2)} - \frac{z - z_2}{z_1 - z_2} \right| \leqslant \eta$$

for all $z \in \mathbf{C}$ with $|z - z_2| < |z_1 - z_2|$. In particular, if g fixes z_1 and z_2 then $|g(z) - z| < \eta |z_1 - z_2|$.

PROPOSITION 2. *There exists $d_2 = d_2(a) \in (0, d_1]$ such that whenever f is a conformal mapping of D_∞ with $\|S_f\|_{D_\infty} \leqslant d_2$ and $f(\infty) = \infty$, then for $j = 1, 2$, $f(\alpha_j)$ is a b-spiral onto $f_j(0)$ with $b \in (1, 2)$.*

PROPOSITION 3. *Given $\varepsilon > 0$ there exists $d_3 = d_3(a, \varepsilon) \in (0, d_1]$ with the following property. If f is a conformal mapping of D_∞ with $\|S_f\|_{D_\infty} \leqslant d_3$ and if f fixes $-1, 1$ and ∞, then $|f_1(0)| < \varepsilon$ and $|f_2(0)| < \varepsilon$.*

PROOF. Let $\eta = \min(1/8, \varepsilon/(5+\varepsilon))$ and choose $d_3 \in (0, d_1]$ so that $(1 - cd_3)^{-2} \leq K_1$ where $c = c(a)$ and $K_1 = K_1(\eta)$ are as in the Remark and Proposition 1.

If g_j is a $K_1^{1/2}$-quasiconformal extension of f_j to $\overline{\mathbf{C}}$ then $g_2^{-1} \circ g_1$ is K_1-quasiconformal in $\overline{\mathbf{C}}$ and fixes each point of $\overline{\Gamma}$. In particular, $g_2^{-1} \circ g_1$ fixes $i, 3i$ and ∞. From Proposition 1 we obtain, with $z_1 = i$ and $z_2 = 3i$,

$$|g_2^{-1}(1) - 1| = |g_2^{-1} \circ g_1(1) - 1| \leq 2\eta \leq 1/4$$

and hence

$$\left| \frac{1 - g_2^{-1}(1)}{-1 - g_2^{-1}(1)} \right| \leq \eta(1-\eta)^{-1} < 1.$$

Since g_2 fixes -1 and ∞, another application of Proposition 1, with $z_1 = -1$ and $z_2 = g_2^{-1}(1)$, yields

$$\left| \frac{g_2(1) - 1}{2} - \frac{1 - g_2^{-1}(1)}{-1 - g_2^{-1}(1)} \right| \leq \eta,$$

and we conclude that

$$|g_2(1) - 1| \leq 2\eta\left(1 + (1-\eta)^{-1}\right). \tag{3.2}$$

Finally, we consider the mapping

$$h(z) = \frac{2g_2(z) - g_2(1) + 1}{g_2(1) + 1},$$

which is K_1-quasiconformal in $\overline{\mathbf{C}}$ and fixes -1, 1 and ∞. Proposition 1 implies $|h(0)| < 2\eta$, so by (3.2) and our choice of η we find

$$|f_2(0)| = |g_2(0)| < 5\eta(1-\eta)^{-1} \leq \varepsilon.$$

Similarly, we find $|f_1(0)| < \varepsilon$.

PROOF OF LEMMA 2. Let $d = \min(d_2(a), d_3(a, 1/5))$ and suppose f is a conformal mapping of D_∞ with $\|S_f\|_{D_\infty} \leq d$. To prove $f_1(0) = f_2(0)$, one argues in exactly the same way as in the proof of Theorem 2 in [**4**], using Propositions 1, 2 and 3 in place of Lemmas 7, 8 and 9.

Now suppose d and f are as above and suppose f fixes -1, -3 and ∞. Let $f(0)$ denote the common value of $f_1(0)$ and $f_2(0)$. Choose η as in Proposition 3 with $\varepsilon = 1/5$ and let g_2 denote a $K_1(\eta)^{1/2}$-quasiconformal extension of f_2. Since $g_2 = f_2$ on $\overline{D}_{\infty,2}$, Proposition 1 implies

$$\left| \frac{f(0) - f(i)}{f_2(-i) - f(i)} - \frac{1}{2} \right| \leq \eta < \frac{1}{6};$$

therefore,

$$|f(i) - f(0)| \geq \frac{1}{3}|f_2(-i) - f(i)|. \tag{3.3}$$

Because g_2 fixes -1, -3 and ∞, two more applications of Proposition 1 yield $|f(i) - i| \leq 2\eta$, $|f_2(-i) + i| \leq 2\eta$. Then $|f_2(-i) - f(i)| \geq 2 - 4\eta > 1$, and (3.1) now follows from (3.3).

4. Proof of Theorem 2. Let d be as in Lemma 2 and let f be a conformal mapping of D with $\|S_f\|_D \leq d$. We may assume $\partial f(D)$ is a Jordan curve, and we will denote the homeomorphic extension of f to $\overline{D}$ by f, as well. We may further assume $f(\infty) = \infty$, so that $\infty \in \partial f(D)$. With these assumptions, in order to show $\partial f(D)$ is not a quasicircle we need only exhibit for each $\lambda > 0$ three points z_1, z_2, z_3 on $\partial D - \{\infty\}$ such that z_2 separates z_1 and z_3 and such that

$$|f(z_2) - f(z_3)| > \lambda |f(z_1) - f(z_3)| \tag{4.1}$$

(see [**1**]).

Fix $\lambda > 0$. We will show that for some m, the following triple on ∂D satisfies (4.1):

$$z_1 = V^m(-\tau_m), \quad z_2 = V^m(i), \quad z_3 = V^m(\tau_m), \tag{4.2}$$

where $V^m(z) = z + 8m$, as before. For this we construct a sequence of conformal mappings f_m from the restrictions of f to the $V^m(D_m)$. We show that the f_m converge to a mapping of D_∞ which, by Lemma 2, nearly preserves the ratio $|z_2 - z_3|/|z_1 - z_3|$. This fact and the nature of the convergence imply (4.1) for the triple (4.2) when m is large.

For each m, we choose U_m to be the Möbius transformation such that

$$f_m(z) = U_m \circ f \circ V^m(z), \qquad z \in D_m,$$

fixes -1, -3 and ∞. Then f_m is a conformal mapping of D_m onto a Jordan domain, and

$$\|S_{f_m}\|_{D_m} = \|S_{f \circ V^m}\|_{D_m} = \|S_f\|_{V^m(D_m)} \leq \|S_f\|_D. \tag{4.3}$$

Because U_m fixes ∞ and preserves cross-ratios, proving (4.1) for the triple (4.2) is equivalent to showing

$$|f_m(i) - f_m(\tau_m)| > \lambda |f_m(-\tau_m) - f_m(\tau_m)|. \tag{4.4}$$

By (4.3), for some fixed K' and all m, $f_m | D_{m,j}$ has a K'-quasiconformal extension $g_{m,j}$ to $\overline{\mathbf{C}}$. The family $\{g_{m,2}\}_{m=1}^\infty$ fixes -1, -3 and ∞ while the family $\{g_{m,2}^{-1} \circ g_{m,1}\}_{m=1}^\infty$ fixes each point of $\overline{\Gamma}$. Thus both families are normal families, and we conclude that there exists an increasing sequence of integers $m(k)$ such that both $\{g_{m(k),1}\}_{k=1}^\infty$ and $\{g_{m(k),2}\}_{k=1}^\infty$ converge uniformly in the chordal metric on $\overline{\mathbf{C}}$ to K'-quasiconformal mappings [**5**]. The limit mappings will be denoted $g_{\infty,1}$ and $g_{\infty,2}$, respectively. The mappings $f_{m(k)}$ of $\overline{D_{m(k)}}$ likewise converge uniformly on compact subsets of D_∞ to the conformal mapping f_∞ of D_∞ satisfying

$$f_\infty | (D_{\infty,j} \cup \Gamma) = g_{\infty,j}, \qquad j = 1, 2.$$

We claim that f_∞ satisfies the hypotheses of both parts of Lemma 2. Clearly f_∞ fixes -1 and -3; moreover, (4.3) implies $\|S_{f_\infty}\|_{D_\infty} \leq d$ since $S_{f_{m(k)}}(z)$ and $\rho_{D_{m(k)}}(z)^{-1}$ converge to $S_{f_\infty}(z)$ and $\rho_{D_\infty}(z)^{-1}$ as k tends to ∞, for $z \in D_\infty$ [**2**]. Consequently, the Remark applies to f_∞, and we deduce that $f_\infty(\infty) = \infty$ and that Lemma 2 is applicable. According to (3.1) we may choose $\mu \in (0, 1/3)$ so that

$$|f_\infty(i) - f_\infty(0)| - \mu > \lambda\mu. \tag{4.5}$$

Next we appeal to the equicontinuity and uniform convergence of the extensions $g_{m(k),j}$. We may first choose $s \in (0,1)$ so that for $z, w \in \overline{\Delta}$ with $|z - w| < s$,

$$\left|g_{m(k),j}(z) - g_{m(k),j}(w)\right| < \mu/4$$

for all k and for $j = 1, 2$. We may then choose k large enough so that $\tau_{m(k)} < s$ and

$$\left|g_{m(k),j}(z) - g_{\infty,j}(z)\right| < \mu/4, \qquad j = 1, 2,$$

whenever $z \in \overline{\Delta}$.

Because $g_{m(k),j} = f_{m(k)}$ on $\overline{D_{m(k),j}}$, (4.4) follows with $m = m(k)$ from (4.5) and the inclusions

$$i, \tau_{m(k)} \in \overline{\Delta} \cap \overline{D}_{m(k),1}, \qquad i, -\tau_{m(k)} \in \overline{\Delta} \cap \overline{D}_{m(k),2}.$$

As we noted, (4.4) is equivalent to (4.1) for the triple (4.2). Since λ was an arbitrary positive number, we conclude that the Jordan curve $\partial f(D)$ is not a K-quasicircle for any K.

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